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Path integral for a Dirac particle in a quantized plane wave field

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Abstract. The problem of a relativistic spinning particle interacting with a quantized electromagnetic plane wave is treated by employing path integral methods and introducing the notion of a coherent state action. The dynamics of the particle spin is described using the supersymmetric action proposed recently by Alexandrou et al. in the so-called global representation. It is shown that to obtain the relative causal Green function, two fermionic identities, related directly to the classical equations of motion, have to be incorporated. The Green function, as constructed, allows us to extract in a natural way the expressions of the corresponding energy spectrum and wave functions.

1 Introduction

Propagators of relativistic particles interacting with intense external fields of various natures provide us with important information about the quantum behavior of these particles. For this purpose, it is still desirable to develop methods for finding the propagators in an external field, based either on solving the equations satisfied by them or on devising their appropriate path integral formulations. However, this latter approach is not without controversy and some problems may be encountered. For instance, the problem of how to describe spin in a path integral has not yet been solved, principally since the classical analog for the internal spin of a particle is not yet available. As is known, the Dirac propagator in an external electromagnetic field is distinguished from the one of a scalar particle by its complicated spinor structure. The problem of its path integral representation has already attracted the attention of researchers a long time ago. Feynman was the first to write a path integral for the probability amplitude in non-relativistic quantum mechanics [1], then a path integral for the scalar particle propagator [2]. He was also the first to attempt to derive a representation for the Dirac propagator via a bosonic path integral [3]. However, to the best of our knowledge his attempt has not known any application in the case of a concrete interaction. Later on, Berezin and Marinov [4] have introduced the integral over Grassmannian variables, where the Dirac propagator has been presented by the path integral over ordinary and over Grassmannian variables which describe the spin degrees of freedom. It should be noted that such representations have formed a basis for developing calculation methods to find propagators for interesting external electromagnetic field configurations. In this respect, Fradkin and Gitman [5], in particular, have recently proposed a straightforward method for describing spin in a path integral and constructing its corresponding relativistic propagator. Their formulation is, up to now, the only successful attempt to describe the spinning point particle in a rigorous fashion. Moreover, the action presented in this method has particularly interesting features owing to the gauge invariance, reparametrization invariance and local supersymmetric form. This method proved most fruitful in finding analytical and exact expressions of the relativistic spin propagators for some configurations of the external fields. In this respect many problems have been solved, such as the case of the interaction with a constant and homogeneous electromagnetic field [6], the case of a plane wave field [7–9], the combination of the two preceding cases [10] and the case of a strong plane wave field, where the anomalous part of the magnetic moment was taken into account [11]. In the present work, which can be regarded as an improvement of a previous one [12], we are revising the problem of the path integral for the quantized electromagnetic field of the one-mode plane wave of the Dirac particle from the point of view of Alexandrou et al. Along this line, we formulate the problem under consideration in the global projection method, which leads, in general, to simpler expressions where the Grassmann proper time and multiplication with the matrix γ_5 do not intervene. On the other hand; the operator projection, which eliminates the superfluous states corresponding to the square of the Dirac operator, shall act at the end of the evolution, and the Dirac operator is written in the path integral representation following [13]. Indeed, as we are proposing here, the action of the relativistic spinning particle arises in a natural manner and the expression exp(iAction) can be integrated directly. This is to be contrasted with [12] where the analogy with the Jaynes–Cummings model was made in order to derive the energy spectrum equations. At least, it is important to note that the elegance and simplicity of evaluating the causal Green function are due essentially to the introduction of two appropriate identities which reduce the number of calculations and lead to the appearance of the classical equations as an argument of the delta functional. On the other hand, the solvability of this problem stems from the properties characterizing the plane wave field.

2 Green function calculations

In this section we consider the interaction of an electron with a circularly polarized one-mode electromagnetic field. Such a field is described by the electromagnetic vector potential A of the form

$$A_\mu(\eta\cdot x) = h(\varepsilon_\mu a \mathrm{e}^{-\mathrm{i}\eta\cdot x} + \varepsilon^*_\mu a^+ \mathrm{e}^{\mathrm{i}\eta\cdot x}), \quad (1)$$

where a, a^+ are boson operators, h the coupling constant, η and ε are respectively the wave vector and the polarization vector given by

$$\eta^\mu = (1,0,0,-1), \quad \varepsilon^\mu = \frac{1}{\sqrt{2}}(0,1,\mathrm{i},0),$$
$$\varepsilon^{\mu *} = \frac{1}{\sqrt{2}}(0,1,-\mathrm{i},0), \quad (2)$$

with the following properties:

$$\varepsilon^\mu\varepsilon_\mu = \varepsilon^{\mu *}\varepsilon^*_\mu = \eta^\mu\eta_\mu = \eta^\mu\varepsilon_\mu = \eta^\mu\varepsilon^*_\mu = 0, \quad \varepsilon^{*\mu}\varepsilon_\mu = -1. \quad (3)$$

Following Schwinger, we present the causal Green function $\widetilde{\mathcal{S}}^{\mathrm{c}}(x,y)$ of a massive spinning particle in an external electromagnetic field $A(x)$ as a matrix element of the operator $\widetilde{\mathcal{S}}^{\mathrm{c}}$:

$$\widetilde{\mathcal{S}}^{\mathrm{c}}(x,y) = \langle x|\,\widetilde{\mathcal{S}}^{\mathrm{c}}\,|y\rangle\,, \quad (4)$$

where $|x\rangle$ is the position eigenstate.

The operator $\widetilde{\mathcal{S}}^{\mathrm{c}}$ can be written via an integral over proper time λ:

$$\widetilde{\mathcal{S}}^{\mathrm{c}} = -\mathrm{i}\int_0^\infty \mathrm{d}\lambda \exp(-\mathrm{i}\mathcal{H}); \quad (5)$$

$\mathcal{H}$ is the Hamiltonian which governs the proper-time evolution

$$\mathcal{H} = -\lambda(\not{\mathcal{P}}^2 - m^2 + \mathrm{i}\epsilon), \quad (6)$$

where $\not{\mathcal{P}} = \mathcal{P}_\mu\gamma^\mu$, $\quad \mathcal{P}_\mu = \mathrm{i}\partial_\mu - gA_\mu(x)$.

In the case under consideration, the Hamiltonian takes the form

$$\mathcal{H} = \lambda\left\{m^2 - \mathcal{P}^2 + \frac{\mathrm{i}g}{2}\mathcal{F}_{\mu\nu}\gamma^\mu\gamma^\nu + \frac{\mathrm{i}}{2}g^2\left[A_\mu, A_\nu\right]\sigma^{\mu\nu}\right\}, \quad (7)$$

with $\sigma^{\mu\nu} = (\mathrm{i}/2)\left[\gamma^\mu,\gamma^\nu\right]$, $\mathcal{F}_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$, so that

$$\mathcal{H} = \lambda\left\{m^2 - \mathcal{P}^2 + \frac{\mathrm{i}g}{2}\mathcal{F}_{\mu\nu}\gamma^\mu\gamma^\nu - \frac{\Omega}{2}(\varepsilon_\mu\varepsilon^*_\nu - \varepsilon_\nu\varepsilon^*_\mu)\gamma^\mu\gamma^\nu\right\}, \quad (8)$$

where $\Omega = g^2h^2$.

Furthermore, as the four-potential A_μ depends only on $\eta\cdot x$ with $\eta^2 = 0$, it is very convenient and natural to choose the Lorentz gauge specified by $\partial_\mu A^\mu = 0$ which is equivalent to $k\cdot A = 0$. This property ensures the solvability of the problem. The scalar product of four-vectors, denoted by a dot, stands for $a\cdot b = a_\mu b^\mu$ and the Minkowski tensor is $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$.

Now we look for the path integral representation for the causal Green function $\widetilde{\mathcal{S}}^{\mathrm{c}}(x,y)$, following the so-called global projection. Within this method, $\widetilde{\mathcal{S}}^{\mathrm{c}}(x,y)$ reads

$$\begin{aligned}
&\widetilde{\mathcal{S}}^{\mathrm{c}}(b,a)\\
&= \frac{1}{2}(\not{\mathcal{P}} + m)\exp\left(\mathrm{i}\gamma^\mu\frac{\partial_l}{\partial\theta^\mu}\right)\int_0^\infty \mathrm{d}e\int DxDpDzDz^*\mathcal{D}\psi\\
&\times\exp\left\{\mathrm{i}\int_0^1\left[\frac{\mathrm{i}}{2}(z^*\,\dot{z} - \dot{z}^*\,z) + p\cdot\dot{x} + \frac{e}{2}(p^2 - m^2)\right.\right.\\
&+ e\sqrt{\Omega}\left[(\varepsilon\cdot p)z\mathrm{e}^{-\mathrm{i}\eta\cdot x} + (\varepsilon^*\cdot p)z^*\mathrm{e}^{\mathrm{i}\eta\cdot x}\right] - e\Omega\left(z^*z + \frac{1}{2}\right)\\
&- 2e\sqrt{\Omega}\left[(\varepsilon\cdot\psi)(\eta\cdot\psi)z\mathrm{e}^{-\mathrm{i}\eta\cdot x} - (\varepsilon^*\cdot\psi)(\eta\cdot\psi)z^*\mathrm{e}^{\mathrm{i}\eta\cdot x}\right]\\
&\left.\left.+ 2e\Omega(\varepsilon^*\cdot\psi)(\varepsilon\cdot\psi) - \mathrm{i}\psi\cdot\dot{\psi}\right]\mathrm{d}\tau + \psi_\mu(1)\psi^\mu(0)\right\}\Bigg|_{\theta=0},
\end{aligned} \quad (9)$$

$$\mathcal{D}\psi = D\psi\left[\int_{\psi(1)+\psi(0)=\theta} D\psi\exp\left\{\int_0^1\psi_\mu\,\dot{\psi}^\mu\,\mathrm{d}\tau\right\}\right]^{-1}, \quad (10)$$

where the integration goes over the trajectories $x_\mu(\tau)$, $p_\mu(\tau)$, $z(\tau)$, $\psi_\mu(\tau)$, parametrized by some parameters $\tau\in[0,1]$, where x and z are bosonic variables describing respectively the external motion of the system and the dynamics of the photons. θ and ψ are fermionic (odd Grassmannian) variables anticommuting with the γ matrices.

The boundary conditions for the coordinate space path integral are

$$x(0) = x_a, \quad x(1) = x_b, \quad z(0) = z_a, \quad z(1) = z_b, \quad (11)$$

and the antiperiodic boundary condition for the spin variable $\psi(\tau)$ is

$$\psi_\mu(0) + \psi_\mu(1) = \theta_\mu. \quad (12)$$

$\psi_\mu(\tau)$ are odd trajectories, anticommuting with γ matrices.

We note also that the classical equation of motion in the framework of the path integral method appears in a natural way and plays an important role since its occurrence guarantees the solvability of the problem. In fact, from the path integral form given by expression (9), we can easily identify the Hamiltonian of the classical system as

$$\begin{aligned}
H &= \frac{e}{2}(p^2 - m^2) + e\sqrt{\Omega}\left[(\varepsilon\cdot p)z\mathrm{e}^{-\mathrm{i}\eta\cdot x} + (\varepsilon^*\cdot p)z^*\mathrm{e}^{\mathrm{i}\eta\cdot x}\right]\\
&- e\Omega\left(z^*z + \frac{1}{2}\right) - 2e\sqrt{\Omega}\Big[(\varepsilon\cdot\psi)(\eta\cdot\psi)z\mathrm{e}^{-\mathrm{i}\eta\cdot x}\\
&- (\varepsilon^*\cdot\psi)(\eta\cdot\psi)z^*\mathrm{e}^{\mathrm{i}\eta\cdot x}\Big] + 2e\Omega(\varepsilon^*\cdot\psi)(\varepsilon\cdot\psi) - \mathrm{i}\psi\cdot\dot{\psi},
\end{aligned}$$

(13)

from which we deduce the equations related to the external motion of the particle:

$$\dot{p}_\mu = -\frac{\partial H}{\partial x^\mu} = \mathrm{i}e\sqrt{\Omega}\Big\{\left[(\varepsilon\cdot p)z\mathrm{e}^{-\mathrm{i}\eta\cdot x} - (\varepsilon^*\cdot p)z^*\mathrm{e}^{\mathrm{i}\eta\cdot x}\right] - 2\left[(\varepsilon\cdot\psi)(\eta\cdot\psi)z\mathrm{e}^{-\mathrm{i}\eta\cdot x} + (\varepsilon^*\cdot\psi)(\eta\cdot\psi)z^*\mathrm{e}^{\mathrm{i}\eta\cdot x}\right]\Big\}\eta_\mu, \tag{14}$$

$$\dot{x}_\mu = -\frac{\partial H}{\partial p^\mu} = -ep_\mu - e\sqrt{\Omega}(z\mathrm{e}^{-\mathrm{i}\eta\cdot x}\varepsilon_\mu + z^*\mathrm{e}^{\mathrm{i}\eta\cdot x}\varepsilon^*_\mu). \tag{15}$$

Projecting these equations on the η_μ direction we get

$$\eta\cdot p = \text{const}, \tag{16}$$

$$-\frac{(\eta\cdot\dot{x})}{e} = \eta\cdot p. \tag{17}$$

As the interaction term is only a function of the product $\eta\cdot x$, let us first introduce the following unitary transformation which frees the interaction term from the position x:

$$z\mathrm{e}^{-\mathrm{i}\eta\cdot x} \to Z, \quad z^*\mathrm{e}^{\mathrm{i}\eta\cdot x} \to Z^*. \tag{18}$$

Using (9) we then find that

$$\begin{aligned}\widetilde{\mathcal{S}}^{\mathrm{c}}(b,a) &= \frac{1}{2}(\not{\mathcal{P}} + m)\exp\left(\mathrm{i}\gamma^\mu\frac{\partial_l}{\partial\theta^\mu}\right)\int_0^\infty \mathrm{d}e \\ &\times\int DxDpDZDZ^*\mathcal{D}\psi\exp\Bigg\{\mathrm{i}\int_0^1\Bigg[\frac{\mathrm{i}}{2}(Z^*\dot{Z} - \dot{Z}^*Z) \\ &+ (p - Z^*Z\eta)\cdot\dot{x} + \frac{e}{2}(p^2 - m^2) \\ &+ e\sqrt{\Omega}\left[(\varepsilon\cdot p)Z + (\varepsilon^*\cdot p)Z^*\right] - e\Omega\left(Z^*Z + \frac{1}{2}\right) \\ &- 2e\sqrt{\Omega}\left[(\varepsilon\cdot\psi)(\eta\cdot\psi)Z - (\varepsilon^*\cdot\psi)(\eta\cdot\psi)Z^*\right] \\ &+ 2e\Omega(\varepsilon^*\cdot\psi)(\varepsilon\cdot\psi) - \mathrm{i}\psi\cdot\dot{\psi}\Bigg]\mathrm{d}\tau + \psi_\mu(1)\psi^\mu(0)\Bigg\}\Bigg|_{\theta=0}.\end{aligned} \tag{19}$$

The external free action can be decoupled from the photon field interaction part by shifting to the new momentum variables $p - Z^*Z\eta \to p$. Taking into account $\varepsilon\cdot\eta = \varepsilon^*\cdot\eta = \eta^2 = 0$, we can write for the Green function $\widetilde{\mathcal{S}}^{\mathrm{c}}(b,a)$:

$$\begin{aligned}\widetilde{\mathcal{S}}^{\mathrm{c}}(b,a) &= \frac{1}{2}(\not{\mathcal{P}} + m)\exp\left(\mathrm{i}\gamma^\mu\frac{\partial_l}{\partial\theta^\mu}\right)\int_0^\infty \mathrm{d}e \\ &\times\int DxDpDZDZ^*\mathcal{D}\psi\exp\Bigg\{\mathrm{i}\int_0^1\Bigg[\frac{\mathrm{i}}{2}(Z^*\dot{Z} - \dot{Z}^*Z) \\ &+ p\cdot\dot{x} + \frac{e}{2}(p^2 - m^2) - \frac{e\Omega}{2} \\ &+ e(\eta\cdot p - \Omega)Z^*Z + e\sqrt{\Omega}\left[(\varepsilon\cdot p)Z + (\varepsilon^*\cdot p)Z^*\right] \\ &- 2e\sqrt{\Omega}\left[(\varepsilon\cdot\psi)(\eta\cdot\psi)Z - (\varepsilon^*\cdot\psi)(\eta\cdot\psi)Z^*\right] \\ &+ 2e\Omega(\varepsilon^*\cdot\psi)(\varepsilon\cdot\psi) - \mathrm{i}\psi\cdot\dot{\psi}\Bigg]\mathrm{d}\tau + \psi_\mu(1)\psi^\mu(0)\Bigg\}\Bigg|_{\theta=0}.\end{aligned} \tag{20}$$

Now it is clear that the free motion present in the evolution is separated from the interaction term and the path integral over x gives a functional δ function $\delta(\dot{p})$ which implies that the momentum p is preserved during the evolution ($p = $ const). This gives

$$\begin{aligned}\widetilde{\mathcal{S}}^{\mathrm{c}}(b,a) &= \frac{1}{2}(\not{\mathcal{P}} + m)\exp(\mathrm{i}\gamma^\mu\frac{\partial_l}{\partial\theta^\mu})\int_0^\infty \mathrm{d}e\int\frac{\mathrm{d}^4p}{(2\pi)^4} \\ &\times\int DZDZ^*\mathcal{D}\psi\exp\Bigg\{\mathrm{i}p\cdot(x_b - x_a) + \frac{\mathrm{i}e}{2}(p^2 - m^2) \\ &- \frac{\mathrm{i}e\Omega}{2} + \mathrm{i}\int_0^1\Bigg[\frac{\mathrm{i}}{2}(Z^*\dot{Z} - \dot{Z}^*Z) + e(\eta\cdot p - \Omega)Z^*Z \\ &+ e\sqrt{\Omega}\left[(\varepsilon\cdot p)Z + (\varepsilon^*\cdot p)Z^*\right] \\ &- 2e\sqrt{\Omega}\left[(\varepsilon\cdot\psi)(\eta\cdot\psi)Z - (\varepsilon^*\cdot\psi)(\eta\cdot\psi)Z^*\right] \\ &+ 2e\Omega(\varepsilon^*\cdot\psi)(\varepsilon\cdot\psi) - \mathrm{i}\psi\cdot\dot{\psi}\Bigg]\mathrm{d}\tau + \psi_\mu(1)\psi^\mu(0)\Bigg\}\Bigg|_{\theta=0}.\end{aligned} \tag{21}$$

We must point out here that according to this constraint the four-dimensional particle motion is reduced to a two-dimensional one described by the variables Z and Z^*. Moreover, the study of the motion can be further simplified by introducing two new variables ζ and ζ' which allow us to consider respectively the variables $\eta\cdot\psi$ and $\varepsilon^*\cdot\psi$ as independent from ψ via the following identities:

$$\int\mathrm{d}\zeta_a\mathrm{d}\zeta_b\delta(\zeta_a - \eta\cdot\psi_a)\int D\zeta Dp_\zeta \times\exp\left[\mathrm{i}\int_0^T p_\zeta(\dot{\zeta} - \eta\cdot\dot{\psi})\right] = 1, \tag{22}$$

$$\int\mathrm{d}\zeta'_a\mathrm{d}\zeta'_b\delta(\zeta'_a - \varepsilon^*\cdot\psi_a)\int D\zeta' Dp_{\zeta'} \times\exp\left[\mathrm{i}\int_0^T p_{\zeta'}(\dot{\zeta}' - \varepsilon^*\cdot\dot{\psi})\right] = 1. \tag{23}$$

Obviously, ζ and ζ' are respectively equal to $\eta\cdot\psi$ and $\varepsilon^*\cdot\psi$ at each time of the evolution and the variables ζ, p_ζ, ζ' and $p_{\zeta'}$ are of the same nature as ψ, i.e., they are odd (Grassmann) variables.

When the identities (22) and (23) are inserted into (21), the Green function takes the following form:

$$\begin{aligned}\widetilde{\mathcal{S}}^{\mathrm{c}}(b,a) &= \frac{1}{2}(\not{\mathcal{P}} + m)\exp\left(\mathrm{i}\gamma^\mu\frac{\partial_l}{\partial\theta^\mu}\right)\int_0^\infty \mathrm{d}e\int\frac{\mathrm{d}^4p}{(2\pi)^4} \\ &\times\int\mathrm{d}\zeta_a\mathrm{d}\zeta_b\delta(\zeta_a - \eta\cdot\psi_a)\int\mathrm{d}\zeta'_a\mathrm{d}\zeta'_b\delta(\zeta'_a - \varepsilon^*\cdot\psi_a) \\ &\times\int D\zeta Dp_\zeta D\zeta' Dp_{\zeta'}DZDZ^*\mathcal{D}\psi\exp\Bigg\{\mathrm{i}p\cdot(x_b - x_a) \\ &+ \frac{\mathrm{i}e}{2}(p^2 - m^2) - \frac{\mathrm{i}e\Omega}{2} \\ &+ \mathrm{i}\int_0^1\Bigg[\frac{\mathrm{i}}{2}(Z^*\dot{Z} - \dot{Z}^*Z) + e(\eta\cdot p - \Omega)Z^*Z \\ &+ e\sqrt{\Omega}\left[(\varepsilon\cdot p)Z + (\varepsilon^*\cdot p)Z^*\right]\end{aligned}$$

$$- 2e\sqrt{\Omega}\left[(\varepsilon\cdot\psi)\zeta Z - \zeta'\zeta Z^*\right] + 2e\Omega\zeta'(\varepsilon\cdot\psi) - \mathrm{i}\psi\cdot\dot{\psi} + p_\zeta(\dot{\zeta} - \eta\cdot\dot{\psi}) + p_{\zeta'}(\dot{\zeta}' - \varepsilon^*\cdot\dot{\psi})\Bigg]\mathrm{d}\tau + \psi_\mu(1)\psi^\mu(0)\Bigg\}\Bigg|_{\theta=0}. \quad (24)$$

We note that due to the antiperiodic character of the spin, the corresponding variables are subjected to a boundary condition involving an additional term $\psi_\mu(1)\psi^\mu(0)$ in the particle action. However, to avoid these difficulties it is sufficient to consider the variable change $\psi(\tau) \to \omega(\tau)$ satisfying

$$\omega_\mu(\tau) = \dot{\psi}_\mu(\tau),$$
$$\psi^\mu(\tau) = \frac{1}{2}\int_0^1 \Delta(\tau - \tau')\omega^\mu(\tau')\mathrm{d}\tau' + \frac{\theta^\mu}{2},$$
$$\Delta(\tau) = \text{sign of}\tau, \quad (25)$$

where the velocity $\omega(\tau)$ keeps the same nature as $\psi(\tau)$. We should add here that the so-called antiperiodic boundary condition at initial and final time is always satisfied: $\psi^\mu(0) + \psi^\mu(1) = \theta^\mu$ without any restrictions on the velocity variables, a fact that can easily be checked. The previous transformation leads to a quadratic term in $\omega(\tau)$ within the action. Hence the Green function becomes

$$\widetilde{S}^{\mathrm{c}}(b,a) = \frac{1}{2}(\not{\mathcal{P}} + m)\exp\left(\mathrm{i}\gamma^\mu\frac{\partial_l}{\partial\theta^\mu}\right)\int_0^\infty \mathrm{d}e\int\frac{\mathrm{d}^4p}{(2\pi)^4} \times\int\mathrm{d}\zeta_a\mathrm{d}\zeta_b\int\mathrm{d}\zeta'_a\mathrm{d}\zeta'_b \times\int D\zeta Dp_\zeta D\zeta' Dp_{\zeta'} DZDZ^* D\omega(\sqrt{\det\Delta})^{-1} \times\delta\left(\zeta_a + \frac{1}{2}\eta\cdot(\omega-\theta)\right)\delta\left(\zeta'_a + \frac{1}{2}\varepsilon^*\cdot(\omega-\theta)\right) \times\exp\Bigg\{\mathrm{i}p\cdot(x_b - x_a) + \frac{\mathrm{i}e}{2}(p^2 - m^2) - \frac{\mathrm{i}e\Omega}{2} + \mathrm{i}\int_0^1\Bigg[\frac{\mathrm{i}}{2}(Z^*\dot{Z} - \dot{Z}^*Z) + e(\eta\cdot p - \Omega)Z^*Z + e\sqrt{\Omega}\left[(\varepsilon\cdot p)Z + (\varepsilon^*\cdot p)Z^*\right] - 2e\sqrt{\Omega}\left[\frac{1}{2}\varepsilon\cdot(\Delta\omega + \theta)\zeta Z - \zeta'\zeta Z^*\right] + e\Omega\zeta'\varepsilon\cdot(\Delta\omega+\theta) + p_\zeta(\dot{\zeta} - \eta\cdot\omega) + p_{\zeta'}(\dot{\zeta}' - \varepsilon^*\cdot\omega) + \frac{\mathrm{i}}{2}\omega\cdot\Delta\omega\Bigg]\mathrm{d}\tau\Bigg\}\Bigg|_{\theta=0}, \quad (26)$$

where we have used the following convolution notation:

$$f\Delta g \equiv \int_0^1\int_0^1 f(\tau)\Delta(\tau - \tau')g(\tau')\mathrm{d}\tau\mathrm{d}\tau'. \quad (27)$$

Now we present the Dirac functions in (26) by means of integral forms over odd (Grassmann) variables:

$$\delta\left(\zeta_a + \frac{1}{2}\eta\cdot(\omega-\theta)\right) = \int\frac{\mathrm{d}p_{\zeta_a}}{2\pi}\exp\left[\mathrm{i}p_{\zeta_a}\left(\zeta_a + \frac{1}{2}\eta\cdot(\omega-\theta)\right)\right], \quad (28)$$

$$\delta\left(\zeta'_a + \frac{1}{2}\varepsilon^*\cdot(\omega-\theta)\right) = \int\frac{\mathrm{d}p_{\zeta'_a}}{2\pi}\exp\left[\mathrm{i}p_{\zeta'_a}\left(\zeta'_a + \frac{1}{2}\varepsilon^*\cdot(\omega-\theta)\right)\right]. \quad (29)$$

In the next step we make the change of variables

$$p_\zeta \to p_\zeta + \frac{1}{2}p_{\zeta a} \quad\text{and}\quad p_{\zeta'} \to p_{\zeta'} + \frac{1}{2}p_{\zeta' a}, \quad (30)$$

to arrive at the result

$$\widetilde{S}^{\mathrm{c}}(b,a) = \frac{1}{2}(\not{\mathcal{P}} + m)\exp\left(\mathrm{i}\gamma^\mu\frac{\partial_l}{\partial\theta^\mu}\right)\int\mathrm{d}e \times\int\frac{\mathrm{d}^4p}{(2\pi)^4}\int\mathrm{d}\zeta_a\mathrm{d}\zeta_b\int\mathrm{d}\zeta'_a\mathrm{d}\zeta'_b\int\frac{\mathrm{d}p_{\zeta_a}}{2\pi}\int\frac{\mathrm{d}p_{\zeta'_a}}{2\pi} \times\int D\zeta Dp_\zeta D\zeta' Dp_{\zeta'} DZDZ^*\exp\Bigg\{\mathrm{i}p\cdot(x_b - x_a) + \frac{\mathrm{i}e}{2}(p^2 - m^2) - \frac{\mathrm{i}e\Omega}{2} + \mathrm{i}\int_0^1\Bigg[\frac{\mathrm{i}}{2}(Z^*\dot{Z} - \dot{Z}^*Z) + e(\eta\cdot p - \Omega)Z^*Z + e\sqrt{\Omega}\left[(\varepsilon\cdot p)Z + (\varepsilon^*\cdot p)Z^*\right] - 2e\sqrt{\Omega}\left[\frac{1}{2}\varepsilon\cdot\theta\zeta Z - \zeta'\zeta Z^*\right] - e\Omega\varepsilon\cdot\theta\zeta' + p_\zeta\dot{\zeta} + p_{\zeta'}\dot{\zeta}' + p_{\zeta_a}\left(\zeta_a + \frac{1}{2}(\dot{\zeta} - \eta\cdot\theta)\right) + p_{\zeta'_a}\left(\zeta'_a + \frac{1}{2}(\dot{\zeta}' - \varepsilon^*\cdot\theta)\right)\Bigg]\mathrm{d}\tau\Bigg\}\Bigg|_{\theta=0} \times\mathcal{I}(\zeta, p_\zeta; \zeta', p_{\zeta'}; Z), \quad (31)$$

where

$$\mathcal{I}(\zeta, p_\zeta; \zeta', p_{\zeta'}; Z) = \int D\omega(\sqrt{\det\Delta})^{-1} \times\exp\left\{-\frac{1}{2}\int_0^1\int_0^1\omega_\mu(\tau)\Delta(\tau-\tau')\omega^\mu(\tau')\mathrm{d}\tau'\mathrm{d}\tau + \int_0^1\mathcal{J}_\mu(\tau)\omega^\mu(\tau)\mathrm{d}\tau\right\}, \quad (32)$$

and the expressions of the external current sources $\mathcal{J}_\mu(\tau)$ present in (32) are defined by

$$\mathcal{J}_\mu(\tau) = -\mathrm{i}p_\zeta(\tau)\eta_\mu - \mathrm{i}p_{\zeta'}(\tau)\varepsilon^*_\mu + \mathrm{i}e\sqrt{\frac{\Omega}{2}}\varepsilon_\mu\int_0^1\zeta(\tau')Z(\tau')\Delta(\tau'-\tau)\mathrm{d}\tau' + \frac{\mathrm{i}e}{2}\Omega\varepsilon_\mu\int_0^1\zeta'(\tau')\Delta(\tau'-\tau)\mathrm{d}\tau'. \quad (33)$$

The main advantage of this step is the possibility it gives us for extracting the classical equation of motion in the evolution. Consequently, the Gaussian integral over the velocities $\omega^\mu(\tau)$ in (32) will be straightforward and the result is simply given by

$$\sqrt{\det \Delta}\exp\left\{-\frac{1}{2}\int_0^1\int_0^1 \mathcal{J}_\mu(\tau)\Delta^{-1}(\tau-\tau')\mathcal{J}^\mu(\tau')\mathrm{d}\tau\mathrm{d}\tau'\right\}$$
$$=\sqrt{\det \Delta} \tag{34}$$
$$\times\exp\left\{\int_0^1\left[-e\Omega p_{\zeta'}(\tau)\zeta'(\tau)-e\sqrt{\Omega}p_{\zeta'}(\tau)\zeta(\tau)Z(\tau)\right]\mathrm{d}\tau\right\}.$$

After substituting the expression (34) into (32) and next into (31), the Green function is reduced to the following form:

$$\begin{aligned}\widetilde{\mathcal{S}}^c(b,a)&=\frac{1}{2}(\not{\mathcal{P}}+m)\exp\left(\mathrm{i}\gamma^\mu\frac{\partial_l}{\partial\theta^\mu}\right)\int_0^\infty\mathrm{d}e\int\frac{\mathrm{d}^4p}{(2\pi)^4}\\&\times\int\mathrm{d}\zeta_a\mathrm{d}\zeta_b\int\mathrm{d}\zeta'_a\mathrm{d}\zeta'_b\int\frac{\mathrm{d}p_{\zeta_a}}{2\pi}\int\frac{\mathrm{d}p_{\zeta'_a}}{2\pi}\\&\times\int D\zeta Dp_\zeta D\zeta' Dp_{\zeta'}DZDZ^*\exp\left\{\mathrm{i}p\cdot(x_b-x_a)\right.\\&+\frac{\mathrm{i}e}{2}(p^2-m^2)-\frac{\mathrm{i}e\Omega}{2}\\&+\mathrm{i}\int_0^1\left[\frac{\mathrm{i}}{2}(Z^*\dot{Z}-\dot{Z}^*Z)+e(\eta\cdot p-\Omega)Z^*Z\right.\\&+e\sqrt{\Omega}\left[(\varepsilon\cdot p)Z+(\varepsilon^*\cdot p)Z^*\right]\\&-2e\sqrt{\Omega}\left[\frac{1}{2}\varepsilon\cdot\theta\zeta Z-\zeta'\zeta Z^*\right]\\&-e\Omega\varepsilon\cdot\theta\zeta'+p_\zeta\dot{\zeta}+p_{\zeta_a}\left(\zeta_a+\frac{1}{2}(\dot{\zeta}-\eta\cdot\theta)\right)\\&+p_{\zeta'}\left(\dot{\zeta}'+\mathrm{i}e\Omega\zeta'+\mathrm{i}e\sqrt{\Omega}\zeta Z\right)\\&\left.\left.+p_{\zeta'_a}\left(\zeta'_a+\frac{1}{2}(\dot{\zeta}'-\varepsilon^*\cdot\theta)\right)\right]\mathrm{d}\tau\right\}\Bigg|_{\theta=0}.\end{aligned} \tag{35}$$

The path integral over p_ζ then gives a functional delta function:

$$\int Dp_\zeta\exp\left(\mathrm{i}\int_0^1 p_\zeta\dot{\zeta}\,\mathrm{d}\tau\right)=\delta(\dot{\zeta}), \tag{36}$$

which can be used to perform the integration over the trajectories $\zeta^\mu(\tau)$ and then enforces the evolution over the latter to satisfy the equation of motion

$$\dot{\zeta}=0; \tag{37}$$

the solution of this equation is trivially found to be

$$\zeta(\tau)=\zeta_a=\text{const.} \tag{38}$$

We notice that (37) is exactly the classical equation of motion for the spin variables derived, as usual, from the Lagrangian and projected over the direction of the wave vector η. In effect, starting from the Lagrangian present in (20), we derive the classical equation of motion

$$-2e(\sqrt{\Omega}\zeta Z+\Omega\zeta')\varepsilon_\mu-2\mathrm{i}\,\dot{\psi}_\mu-\dot{p}_\zeta\,\eta_\mu-\dot{p}_{\zeta'}\,\varepsilon^*_\mu=0. \tag{39}$$

Next multiplying by η^μ, we easily get

$$-2\mathrm{i}\eta\cdot\dot{\psi}=0. \tag{40}$$

That is to say, we again obtain the same argument of the delta function ($\dot{\zeta}=0$).

By inserting this result in (35), we obtain the following expression for the Green function:

$$\begin{aligned}\widetilde{\mathcal{S}}^c(b,a)&=\frac{1}{2}(\not{\mathcal{P}}+m)\exp\left(\mathrm{i}\gamma^\mu\frac{\partial_l}{\partial\theta^\mu}\right)\int_0^\infty\mathrm{d}e\int\frac{\mathrm{d}^4p}{(2\pi)^4}\\&\times\int\mathrm{d}\zeta_a\mathrm{d}\zeta_b\delta(\zeta_b-\zeta_a)\int\mathrm{d}\zeta'_a\mathrm{d}\zeta'_b\\&\times\int\frac{\mathrm{d}p_{\zeta_a}}{2\pi}\int\frac{\mathrm{d}p_{\zeta'_a}}{2\pi}D\zeta' Dp_{\zeta'}DZDZ^*\\&\times\exp\left\{\mathrm{i}p\cdot(x_b-x_a)+\frac{\mathrm{i}e}{2}(p^2-m^2)-\frac{\mathrm{i}e\Omega}{2}\right.\\&+\mathrm{i}\int_0^1\left[\frac{\mathrm{i}}{2}(Z^*\dot{Z}-\dot{Z}^*Z)+e(\eta\cdot p-\Omega)Z^*Z\right.\\&+e\sqrt{\Omega}\left[(\varepsilon\cdot p)Z+(\varepsilon^*\cdot p)Z^*\right]\\&-2e\sqrt{\Omega}\left[\frac{1}{2}\varepsilon\cdot\theta\zeta_a Z-\zeta'\zeta_a Z^*\right]\\&-e\Omega\varepsilon\cdot\theta\zeta'+p_{\zeta_a}\left(\zeta_a-\frac{1}{2}\eta\cdot\theta\right)\\&+p_{\zeta'}\left(\dot{\zeta}'+\mathrm{i}e\Omega\zeta'+\mathrm{i}e\sqrt{\Omega}\zeta_a Z\right)\\&\left.\left.+p_{\zeta'_a}\left(\frac{1}{2}(\zeta'_b+\zeta'_a-\varepsilon^*\cdot\theta)\right)\right]\mathrm{d}\tau\right\}\Bigg|_{\theta=0}.\end{aligned} \tag{41}$$

At this level, let us consider the integration over the complex variables Z. To this end, we can put the part of $\widetilde{\mathcal{S}}^c(b,a)$ depending on Z in the form

$$\int DZDZ^*\exp\left\{\mathrm{i}\int_0^1\left[\frac{\mathrm{i}}{2}(Z^*\dot{Z}-\dot{Z}^*Z)-wZ^*Z-I(\tau)Z-Z^*J(\tau)\right]\mathrm{d}\tau\right\}, \tag{42}$$

where

$$\begin{aligned}I(\tau)&=-e\sqrt{\Omega}\left[\varepsilon\cdot p-\varepsilon\cdot\theta\zeta_a+\mathrm{i}p_{\zeta'}\zeta_a\right],\\J(\tau)&=-e\sqrt{\Omega}\left[\varepsilon^*\cdot p+2\zeta'\zeta_a\right],\quad w=e\omega,\end{aligned} \tag{43}$$

and $\omega=(\Omega-\eta\cdot p)$.

Expression (42) represents the path integral for the forced harmonic oscillator in the bosonic coherent-state representation. This can be integrated, in the usual way to obtain the familiar result

$$\begin{aligned}&\exp\left[-\frac{|Z_b|^2+|Z_a|^2}{2}\right]\\&\times\exp\left\{Z_b^*\mathrm{e}^{-\mathrm{i}e\omega}Z_a-\mathrm{i}Z_a\int_0^1\mathrm{d}\tau\mathrm{e}^{-\mathrm{i}e\omega\tau}I(\tau)\right.\\&-\mathrm{i}Z_b^*\int_0^1\mathrm{e}^{-\mathrm{i}e\omega(1-\tau)}J(\tau)\mathrm{d}\tau\end{aligned}$$

$$- \int_0^1 \int_0^\tau I(\tau) \mathrm{e}^{-\mathrm{i}e\omega(\tau-\tau')} J(\tau') \mathrm{d}\tau' \mathrm{d}\tau \Big\} . \quad (44)$$

So, after some calculations the Green function becomes

$$\begin{aligned}
\widetilde{\mathcal{S}}^{\mathrm{c}}(b,a) &= \frac{1}{2}(\not{\mathcal{P}} + m) \exp\left(\mathrm{i}\gamma^\mu \frac{\partial_l}{\partial \theta^\mu}\right) \int_0^\infty \mathrm{d}e \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \\
&\times \int \mathrm{d}\zeta_a \mathrm{d}\zeta_b \delta(\zeta_b - \zeta_a) \int \mathrm{d}\zeta'_a \mathrm{d}\zeta'_b \int \frac{\mathrm{d}p_{\zeta_a}}{2\pi} \int \frac{\mathrm{d}p_{\zeta'_a}}{2\pi} \\
&\times \int D\zeta' D p_{\zeta'} \exp\Big\{ \mathrm{i}p\cdot(x_b - x_a) + \frac{\mathrm{i}e}{2}(p^2 - m^2) \\
&- \frac{\mathrm{i}e\Omega}{2} - \frac{|Z_b|^2 + |Z_a|^2}{2} + Z_b^* \mathrm{e}^{-\mathrm{i}e\omega} Z_a \\
&+ \frac{\sqrt{\Omega}}{\omega}(1 - \mathrm{e}^{-\mathrm{i}e\omega})(\varepsilon\cdot p - \varepsilon\cdot\theta\zeta_a) Z_a \\
&+ \frac{\sqrt{\Omega}}{\omega}(1 - \mathrm{e}^{-\mathrm{i}e\omega}) Z_b^* \varepsilon^*\cdot p \\
&+ \frac{\mathrm{i}e\Omega}{\omega}\left(1 + \frac{\mathrm{i}}{e\omega}(1 - \mathrm{e}^{-\mathrm{i}e\omega})\right)((\varepsilon^*\cdot p)(\varepsilon\cdot p) \\
&- (\varepsilon^*\cdot p)(\varepsilon\cdot\theta)\zeta_a) \\
&+ \mathrm{i}\int_0^1 \Big[-e\Omega\varepsilon\cdot\theta\zeta' + p_{\zeta_a}\left(\zeta_a - \frac{1}{2}\eta\cdot\theta\right) \\
&+ p_{\zeta'_a}\left(\frac{1}{2}(\zeta'_b + \zeta'_a - \varepsilon^*\cdot\theta)\right) \\
&- 2e\sqrt{\Omega}\mathrm{e}^{-\mathrm{i}e\omega} Z_b^* \mathrm{e}^{\mathrm{i}e\omega\tau}\zeta_a\zeta' \\
&- 2\mathrm{i}e^2\Omega\varepsilon\cdot p\int_0^\tau \mathrm{e}^{-\mathrm{i}e\omega(\tau-\tau')}\zeta_a\zeta'(\tau')\mathrm{d}\tau' \\
&+ p_{\zeta'}\Big(\dot{\zeta}' + \mathrm{i}e\Omega\zeta' + \mathrm{i}e\sqrt{\Omega}\mathrm{e}^{-\mathrm{i}e\omega\tau}\zeta_a Z_a \\
&+ \frac{\mathrm{i}e\Omega}{\omega}(1 - \mathrm{e}^{-\mathrm{i}e\omega\tau})\varepsilon^*\cdot p\zeta_a\Big)\Big] \mathrm{d}\tau \Big\}\Big|_{\theta=0} . \quad (45)
\end{aligned}$$

Next, we integrate over p_{ζ_a} and $p_{\zeta'_a}$ to get, respectively,

$$\int \frac{\mathrm{d}p_{\zeta_a}}{2\pi} \exp\left[\mathrm{i}p_{\zeta_a}\left(\zeta_a - \frac{1}{2}\eta\cdot\theta\right)\right] = \delta(\zeta_a - \frac{1}{2}\eta\cdot\theta), \quad (46)$$

$$\begin{aligned}
&\int \frac{\mathrm{d}p_{\zeta'_a}}{2\pi} \exp\left[\mathrm{i}\int_0^1 p_{\zeta'_a}\left(\frac{1}{2}(\zeta'_b + \zeta'_a - \varepsilon^*\cdot\theta)\right)\right] \\
&= \delta\left(\frac{1}{2}(\zeta'_b + \zeta'_a - \varepsilon^*\cdot\theta)\right). \quad (47)
\end{aligned}$$

In other words, the following constraints are imposed:

$$\zeta_a = \frac{1}{2}\eta\cdot\theta, \quad (48)$$

$$\zeta'_b + \zeta'_a = \varepsilon^*\cdot\theta. \quad (49)$$

From (38) and (48), it is obviously deduced that the following condition is also satisfied:

$$\zeta_b + \zeta_a = \eta\cdot\theta. \quad (50)$$

Equations (49) and (50) signify that the boundary condition for the spin variables is preserved.

With this result substituted into (45), the functional integration over $p_{\zeta'}$ produces a delta functional

$$\begin{aligned}
&\int Dp_{\zeta'} \exp\Big(\mathrm{i}\int_0^1 p_{\zeta'}\Big(\dot{\zeta}' + \mathrm{i}e\Omega\zeta' + \frac{\mathrm{i}e}{2}\sqrt{\Omega}\mathrm{e}^{-\mathrm{i}e\omega\tau} Z_a\eta\cdot\theta \\
&+ \frac{\mathrm{i}e\Omega}{2\omega}(1 - \mathrm{e}^{-\mathrm{i}e\omega\tau})(\varepsilon^*\cdot p)(\eta\cdot\theta)\Big)\mathrm{d}\tau\Big) \\
&= \delta\Big(\dot{\zeta}' + \mathrm{i}e\Omega\zeta' + \frac{\mathrm{i}e}{2}\sqrt{\Omega}\mathrm{e}^{-\mathrm{i}e\omega\tau} Z_a\eta\cdot\theta \\
&+ \frac{\mathrm{i}e\Omega}{2\omega}(1 - \mathrm{e}^{-\mathrm{i}e\omega\tau})(\varepsilon^*\cdot p)(\eta\cdot\theta)\Big). \quad (51)
\end{aligned}$$

This means that the evolution over ζ' corresponds only to the following equation of motion:

$$\begin{aligned}
&\dot{\zeta}' + \mathrm{i}e\Omega\zeta' + \frac{\mathrm{i}e}{2}\sqrt{\Omega}\mathrm{e}^{-\mathrm{i}e\omega\tau} Z_a\eta\cdot\theta \\
&+ \frac{\mathrm{i}e\Omega}{2\omega}(1 - \mathrm{e}^{-\mathrm{i}e\omega\tau})(\varepsilon^*\cdot p)(\eta\cdot\theta) = 0. \quad (52)
\end{aligned}$$

We note that (52) is nothing but the classical equation of motion for the spin variables projected over the direction of the complex conjugate to the polarization vector. This allows the functional integration to be performed over ζ' by substituting ζ' with the solution to the equation in the argument of the delta functional. Therefore, after some straightforward calculations, one delta function remains:

$$\begin{aligned}
&\delta\Big(\zeta'_b - \mathrm{e}^{-\mathrm{i}e\Omega}\zeta'_a - \frac{\sqrt{\Omega}}{2}\frac{(\mathrm{e}^{-\mathrm{i}e\omega} - \mathrm{e}^{-\mathrm{i}e\Omega})}{(\omega - \Omega)}\eta\cdot\theta Z_a \\
&+ \frac{\Omega}{2\omega}\left(\frac{(1 - \mathrm{e}^{-\mathrm{i}e\Omega})}{\Omega} + \frac{(\mathrm{e}^{-\mathrm{i}e\omega} - \mathrm{e}^{-\mathrm{i}e\Omega})}{(\omega - \Omega)}\right) \\
&\times (\varepsilon^*\cdot p)(\eta\cdot p)\Big). \quad (53)
\end{aligned}$$

Now, we integrate over the endpoint ζ'_b of the ζ' trajectories and then we use the delta function $\delta((1/2)(\zeta'_b + \zeta'_a - \varepsilon^*\cdot\theta))$ in order to fix the initial boundary condition for the solution $\zeta'(\tau)$ to (52). For this purpose we use the following well-known property of the delta function $\delta(a\zeta + b) = a\delta(\zeta + (b/a))$ valid for an odd Grassmann variable. We thus easily obtain

$$\begin{aligned}
\zeta'_a &= \frac{\sqrt{\Omega}}{2}\frac{(\mathrm{e}^{-\mathrm{i}e\Omega} - \mathrm{e}^{-\mathrm{i}e\omega})}{(\omega - \Omega)(1 + \mathrm{e}^{-\mathrm{i}e\Omega})}\eta\cdot\theta Z_a + \frac{\varepsilon^*\cdot\theta}{1 + \mathrm{e}^{-\mathrm{i}e\Omega}} \quad (54) \\
&- \frac{1}{2\omega}\frac{\left(\Omega(1 - \mathrm{e}^{-\mathrm{i}e\omega}) - \omega(1 - \mathrm{e}^{-\mathrm{i}e\Omega})\right)}{(\omega - \Omega)(1 + \mathrm{e}^{-\mathrm{i}e\Omega})}(\varepsilon^*\cdot p)(\eta\cdot\theta).
\end{aligned}$$

Then, after replacing ζ'_a by the right-hand side of this last equation, the causal Green function (45) can be written as

$$\begin{aligned}
\widetilde{\mathcal{S}}^{\mathrm{c}}(b,a) &= \frac{1}{2}(\not{\mathcal{P}} + m) \exp\left(\mathrm{i}\gamma^\mu \frac{\partial_l}{\partial \theta^\mu}\right) \int_0^\infty \mathrm{d}e \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \\
&\times \frac{1}{2}(1 + \mathrm{e}^{-\mathrm{i}e\Omega}) \exp\Big\{ \mathrm{i}p\cdot(x_b - x_a)
\end{aligned}$$

$$+ \frac{\mathrm{i}e}{2}(p^2-m^2) - \frac{\mathrm{i}e\Omega}{2} - \frac{|Z_b|^2 + |Z_a|^2}{2} + Z_b^* \mathrm{e}^{-\mathrm{i}e\omega} Z_a$$
$$+ \sqrt{\Omega}\left(\frac{1}{\omega}(1-\mathrm{e}^{-\mathrm{i}e\omega})\varepsilon\cdot p\right.$$
$$\left. + \frac{(\mathrm{e}^{-\mathrm{i}e\omega} - \mathrm{e}^{-\mathrm{i}e\Omega})}{(\omega-\Omega)(1+\mathrm{e}^{-\mathrm{i}e\Omega})}(\varepsilon\cdot\theta)(\eta\cdot\theta)\right) Z_a$$
$$+ \sqrt{\Omega}\left(\frac{1}{\omega}(1-\mathrm{e}^{-\mathrm{i}e\omega})\varepsilon^*\cdot p\right.$$
$$\left. + \frac{(\mathrm{e}^{-\mathrm{i}e\omega} - \mathrm{e}^{-\mathrm{i}e\Omega})}{(\omega-\Omega)(1+\mathrm{e}^{-\mathrm{i}e\Omega})}(\eta\cdot\theta)(\varepsilon^*\cdot\theta)\right) Z_b^*$$
$$+ \frac{\mathrm{i}e\Omega}{\omega}\left(1 + \frac{\mathrm{i}}{e\omega}(1-\mathrm{e}^{-\mathrm{i}e\omega})\right)(\varepsilon^*\cdot p)(\varepsilon\cdot p)$$
$$- \frac{(1-\mathrm{e}^{-\mathrm{i}e\Omega})}{(1+\mathrm{e}^{-\mathrm{i}e\Omega})}(\varepsilon\cdot\theta)(\varepsilon^*\cdot\theta)$$
$$- \frac{\left(\omega(1-\mathrm{e}^{-\mathrm{i}e\Omega}) - \Omega(1-\mathrm{e}^{-\mathrm{i}e\omega})\right)}{\omega(\omega-\Omega)(1+\mathrm{e}^{-\mathrm{i}e\Omega})}(\varepsilon^*\cdot p)(\varepsilon\cdot\theta)(\eta\cdot\theta)$$
$$- \frac{\left(\omega(1-\mathrm{e}^{-\mathrm{i}e\Omega}) - \Omega(1-\mathrm{e}^{-\mathrm{i}e\omega})\right)}{\omega(\omega-\Omega)(1+\mathrm{e}^{-\mathrm{i}e\Omega})}$$
$$\times (\varepsilon\cdot p)(\eta\cdot\theta)(\varepsilon^*\cdot\theta)\Bigg\}\Bigg|_{\theta=0}. \quad (55)$$

Before returning to the old variables z_b, z_a, it is convenient to rewrite the matrix Green function as

$$\widetilde{\mathcal{S}}^{\mathrm{c}}(x_b,x_a) = \int \frac{\mathrm{d}z_b^*\mathrm{d}z_b}{2\mathrm{i}\pi}\frac{\mathrm{d}z_a^*\mathrm{d}z_a}{2\mathrm{i}\pi}\,|z_b\rangle\,\widetilde{\mathcal{S}}^{\mathrm{c}}(b,a)\,\langle z_a|\,; \quad (56)$$

or by using the occupation number basis we write

$$\widetilde{\mathcal{S}}^{\mathrm{c}}(x_b,x_a) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\int \frac{\mathrm{d}z_b^*\mathrm{d}z_b}{2i\pi}\frac{\mathrm{d}z_a^*\mathrm{d}z_a}{2i\pi}\frac{(z_b)^{n_1}}{\sqrt{n_1!}}\frac{(z_a^*)^{n_2}}{\sqrt{n_2!}}\,|\,n_1\rangle \times \widetilde{\mathbb{S}}^{\mathrm{c}}(b,a)\langle n_2\Big|, \quad (57)$$

with

$$\widetilde{\mathbb{S}}^{\mathrm{c}}(b,a) = \exp\left(-\frac{|z_b|^2}{2} - \frac{|z_a|^2}{2}\right)\widetilde{\mathcal{S}}^{\mathrm{c}}(b,a). \quad (58)$$

By the change of variable

$$Z \to Z + \frac{\sqrt{\Omega}}{\omega}\varepsilon^*\cdot p, \quad \text{with } Z = z\mathrm{e}^{-\mathrm{i}\eta\cdot x}, \quad (59)$$

carried out only on the points z_b^* and z_a, it is easy to see that the expression for the Green function can be put in the form

$$\widetilde{\mathbb{S}}^{\mathrm{c}}(b,a) = \frac{1}{2}(\not{\mathcal{P}}+m)\exp\left(\mathrm{i}\gamma^\mu\frac{\partial_l}{\partial\theta^\mu}\right)\int_0^\infty \mathrm{d}e\int\frac{\mathrm{d}^4p}{(2\pi)^4}$$
$$\times\frac{1}{2}(1+\mathrm{e}^{-\mathrm{i}e\Omega})\exp\Bigg\{\mathrm{i}p\cdot(x_b-x_a)$$
$$+ \frac{\mathrm{i}e}{2}(p^2-m^2) - \frac{\mathrm{i}e\Omega}{2} + \frac{\mathrm{i}e\Omega}{\omega}(\varepsilon\cdot p)(\varepsilon^*\cdot p)$$
$$+ \frac{\Omega}{\omega^2}(\varepsilon\cdot p)(\varepsilon^*\cdot p) + z_b^*\mathrm{e}^{-\mathrm{i}e\omega}\mathrm{e}^{\mathrm{i}\eta\cdot(x_b-x_a)}z_a$$
$$+ \left(\frac{\sqrt{\Omega}\varepsilon\cdot p}{\omega}\mathrm{e}^{-\mathrm{i}\eta\cdot x_a}z_a - \frac{\sqrt{\Omega}\varepsilon^*\cdot p}{\omega}\mathrm{e}^{\mathrm{i}\eta\cdot x_a}z_a^*\right)$$
$$+ \left(\frac{\sqrt{\Omega}\varepsilon^*\cdot p}{\omega}\mathrm{e}^{\mathrm{i}\eta\cdot x_b}z_b^* - \frac{\sqrt{\Omega}\varepsilon\cdot p}{\omega}\mathrm{e}^{-\mathrm{i}\eta\cdot x_b}z_b\right)$$
$$- |z_b|^2 - |z_a|^2$$
$$+ \left[\frac{\sqrt{\Omega}(\mathrm{e}^{-\mathrm{i}e\omega}-\mathrm{e}^{-\mathrm{i}e\Omega})z_a\mathrm{e}^{-\mathrm{i}\eta\cdot x_a}}{(\omega-\Omega)(1+\mathrm{e}^{-\mathrm{i}e\Omega})} - \frac{(1-\mathrm{e}^{-\mathrm{i}e\Omega})\varepsilon^*\cdot p}{\omega(1+\mathrm{e}^{-\mathrm{i}e\Omega})}\right]$$
$$\times (\varepsilon\cdot\theta)(\eta\cdot\theta)$$
$$+ \left[\frac{\sqrt{\Omega}(\mathrm{e}^{-\mathrm{i}e\omega}-\mathrm{e}^{-\mathrm{i}e\Omega})z_b^*\mathrm{e}^{\mathrm{i}\eta\cdot x_b}}{(\omega-\Omega)(1+\mathrm{e}^{-\mathrm{i}e\Omega})} - \frac{(1-\mathrm{e}^{-\mathrm{i}e\Omega})\varepsilon\cdot p}{\omega(1+\mathrm{e}^{-\mathrm{i}e\Omega})}\right]$$
$$\times (\eta\cdot\theta)(\varepsilon^*\cdot\theta)$$
$$- \frac{(1-\mathrm{e}^{-\mathrm{i}e\Omega})}{(1+\mathrm{e}^{-\mathrm{i}e\Omega})}(\varepsilon\cdot\theta)(\varepsilon^*\cdot\theta)\Bigg\}\Bigg|_{\theta=0}. \quad (60)$$

Now we perform the differentiation with respect to θ by means of the identities

$$\exp\left(\mathrm{i}\gamma^\mu\frac{\partial_l}{\partial\theta^\mu}\right)f(\theta)\Big|_{\theta=0} = f\left(\frac{\partial_l}{\partial\zeta}\right)\exp\left(\mathrm{i}\gamma_\mu\zeta^\mu\right)\Big|_{\theta=0}, \quad (61)$$

$$\exp(\mathrm{i}\gamma_\mu\zeta^\mu) = 1 + \mathrm{i}\gamma_\mu\zeta^\mu - \frac{1}{2}\zeta^\mu\zeta^\nu\gamma_\mu\gamma_\nu + \frac{\mathrm{i}}{6}\zeta^\mu\zeta^\nu\zeta^\sigma\gamma_\mu\gamma_\nu\gamma_\sigma + \zeta^0\zeta^1\zeta^2\zeta^3\gamma^5. \quad (62)$$

Thus, after some straightforward calculations and after expanding $\exp\left(z_b^*\mathrm{e}^{-\mathrm{i}e\omega}\mathrm{e}^{\mathrm{i}\eta\cdot(x_b-x_a)}z_a\right)$ as a series, the Green function becomes

$$\widetilde{\mathbb{S}}^{\mathrm{c}}(b,a) = \frac{1}{2}(\not{\mathcal{P}}+m)\sum_{n=0}^{\infty}\int_0^\infty \mathrm{d}e\int\frac{\mathrm{d}^4p}{(2\pi)^4}$$
$$\times\exp\Bigg\{\mathrm{i}(p+n\eta)\cdot(x_b-x_a) + \frac{\mathrm{i}e}{2}(p^2-m^2)$$
$$+ \mathrm{i}ne\eta\cdot p - \mathrm{i}(n+\frac{1}{2})e\Omega + \frac{\mathrm{i}e\Omega}{\omega}(\varepsilon\cdot p)(\varepsilon^*\cdot p)$$
$$+ \frac{\Omega}{\omega^2}(\varepsilon\cdot p)(\varepsilon^*\cdot p)$$
$$+ \left(\frac{\sqrt{\Omega}\varepsilon\cdot p}{\omega}\mathrm{e}^{-\mathrm{i}\eta\cdot x_a}z_a - \frac{\sqrt{\Omega}\varepsilon^*\cdot p}{\omega}\mathrm{e}^{\mathrm{i}\eta\cdot x_a}z_a^*\right)$$
$$+ \left(\frac{\sqrt{\Omega}\varepsilon^*\cdot p}{\omega}\mathrm{e}^{\mathrm{i}\eta\cdot x_b}z_b^* - \frac{\sqrt{\Omega}\varepsilon\cdot p}{\omega}\mathrm{e}^{-\mathrm{i}\eta\cdot x_b}z_b\right)$$
$$- |z_b|^2 - |z_a|^2\Bigg\}$$
$$\times\Bigg\{\mathrm{e}^{-\mathrm{i}e\Omega/2}\left(\cos\left(\frac{e\Omega}{2}\right) - \mathrm{i}\sin\left(\frac{e\Omega}{2}\right)S\right)$$
$$- \left[\frac{\sqrt{\Omega}(\mathrm{e}^{-\mathrm{i}e\omega}-\mathrm{e}^{-\mathrm{i}e\Omega})z_a\mathrm{e}^{-\mathrm{i}\eta\cdot x_a}}{2(\omega-\Omega)} - \frac{(1-\mathrm{e}^{-\mathrm{i}e\Omega})\varepsilon^*\cdot p}{2\omega}\right]$$

$$\times \not\varepsilon \not\eta + \left[\frac{\sqrt{\Omega}(\mathrm{e}^{-\mathrm{i}e\omega} - \mathrm{e}^{-\mathrm{i}e\Omega}) z_b^* \mathrm{e}^{\mathrm{i}\eta\cdot x_b}}{2(\omega - \Omega)} - \frac{(1 - \mathrm{e}^{-\mathrm{i}e\Omega})\varepsilon \cdot p}{2\omega}\right] \times \not\varepsilon^* \not\eta\Big\} \frac{(z_b^*)^n (z_a)^n}{n!}, \tag{63}$$

where $S = 1 + \not\varepsilon \not\varepsilon^*$ is the spin operator with eigenvalues $s = \pm 1$.

To extract the energy spectrum, a shift in the momentum variables $p \to p - \eta$ can be judiciously made in several terms of the expression (63). The transformed Green function then becomes

$$\begin{aligned}\widetilde{\mathbb{S}}^c(b,a) = \frac{1}{2}(\not P + m) \sum_{n=0}^{\infty} \int \mathrm{d}e \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \\ \times \exp\Big\{ \mathrm{i}(p + n\eta)\cdot(x_b - x_a) + \frac{\mathrm{i}e}{2}(p^2 - m^2) \\ + \mathrm{i}en\eta\cdot p - \mathrm{i}e(n+1)\Omega \\ + \frac{\mathrm{i}e\Omega}{\omega}(\varepsilon\cdot p)(\varepsilon^*\cdot p) + \frac{\Omega}{\omega^2}(\varepsilon\cdot p)(\varepsilon^*\cdot p) \\ + \left(\frac{\sqrt{\Omega}\varepsilon\cdot p}{\omega}\mathrm{e}^{-\mathrm{i}\eta\cdot x_a} z_a - \frac{\sqrt{\Omega}\varepsilon^*\cdot p}{\omega}\mathrm{e}^{\mathrm{i}\eta\cdot x_a} z_a^*\right) \\ + \left(\frac{\sqrt{\Omega}\varepsilon^*\cdot p}{\omega}\mathrm{e}^{\mathrm{i}\eta\cdot x_b} z_b^* - \frac{\sqrt{\Omega}\varepsilon\cdot p}{\omega}\mathrm{e}^{-\mathrm{i}\eta\cdot x_b} z_b\right) \\ - |z_b|^2 - |z_a|^2\Big\} \frac{(z_b^*)^n (z_a)^n}{n!} \\ \times \Big\{ \mathrm{e}^{-\mathrm{i}e\Omega/2}\Big[\frac{1}{2}(1+S) \\ + \left(\frac{\sqrt{\Omega}}{2\eta\cdot p}(\mathrm{e}^{-\mathrm{i}\eta\cdot x_b} - \mathrm{e}^{-\mathrm{i}\eta\cdot xa}) z_a - \frac{\varepsilon^*\cdot p}{2\omega}\right) \not\varepsilon \not\eta \\ + \left(\frac{\sqrt{\Omega}}{2\eta\cdot p}(\mathrm{e}^{\mathrm{i}\eta\cdot x_b} - \mathrm{e}^{\mathrm{i}\eta\cdot xa}) z_b^* + \frac{\varepsilon\cdot p}{2\omega}\right) \not\varepsilon^* \not\eta\Big] \\ + \mathrm{e}^{\mathrm{i}e\Omega/2} \\ \times \left[\frac{1}{2}(1-S) + \frac{\varepsilon^*\cdot p}{2\omega} \not\varepsilon \not\eta - \frac{\varepsilon\cdot p}{2\omega} \not\varepsilon^* \not\eta\right]\Big\}.\end{aligned} \tag{64}$$

It is now convenient to carry out the integration over e, yielding

$$\begin{aligned}\widetilde{\mathbb{S}}^c(b,a) = (\mathrm{i})(\not P + m) \sum_{n=0}^{\infty} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \\ \times \exp\Big\{ \mathrm{i}(p + n\eta)\cdot(x_b - x_a) + \frac{\Omega}{\omega^2}(\varepsilon\cdot p)(\varepsilon^*\cdot p) \\ + \left(\frac{\sqrt{\Omega}\varepsilon\cdot p}{\omega}\mathrm{e}^{-\mathrm{i}\eta\cdot x_a} z_a - \frac{\sqrt{\Omega}\varepsilon^*\cdot p}{\omega}\mathrm{e}^{\mathrm{i}\eta\cdot x_a} z_a^*\right) \\ + \left(\frac{\sqrt{\Omega}\varepsilon^*\cdot p}{\omega}\mathrm{e}^{\mathrm{i}\eta\cdot x_b} z_b^* - \frac{\sqrt{\Omega}\varepsilon\cdot p}{\omega}\mathrm{e}^{-\mathrm{i}\eta\cdot x_b} z_b\right) \\ - |z_b|^2 - |z_a|^2\Big\} \frac{(z_b^*)^n (z_a)^n}{n!} \\ \times \Big\{ \frac{1}{d_{p,n}^-}\Big[\frac{1}{2}(1+S) \\ + \left(\frac{\sqrt{\Omega}}{2\eta\cdot p}(\mathrm{e}^{-\mathrm{i}\eta\cdot x_b} - \mathrm{e}^{-\mathrm{i}\eta\cdot xa}) z_a - \frac{\varepsilon^*\cdot p}{2\omega}\right) \not\varepsilon \not\eta \\ + \left(\frac{\sqrt{\Omega}}{2\eta\cdot p}(\mathrm{e}^{\mathrm{i}\eta\cdot x_b} - \mathrm{e}^{\mathrm{i}\eta\cdot xa}) z_b^* + \frac{\varepsilon\cdot p}{2\omega}\right) \not\varepsilon^* \not\eta\Big] \\ + \frac{1}{d_{p,n}^+}\left[\frac{1}{2}(1-S) + \frac{\varepsilon^*\cdot p}{2\omega} \not\varepsilon \not\eta - \frac{\varepsilon\cdot p}{2\omega} \not\varepsilon^* \not\eta\right]\Big\},\end{aligned} \tag{65}$$

and

$$d_{p,n}^{\mp} = p^2 + 2n\eta\cdot p - 2(n+1)\Omega + \frac{2\Omega}{\Omega - \eta\cdot p}(\varepsilon^*\cdot p)(\varepsilon\cdot p) - m^2 \mp \Omega. \tag{66}$$

Therefore by changing p into $(-p)$ in the above relation (66) the bound states energy levels are determined by the equation

$$p^2 = 2n\eta\cdot p + 2(n+1)\Omega - \frac{2\Omega}{\Omega + \eta\cdot p}(\varepsilon^*\cdot p)(\varepsilon\cdot p) + m^2 \pm \Omega. \tag{67}$$

In order to pass to the mass shell representation, it is convenient to consider a new momentum vector P which is on the mass shell and is connected with the previous momentum vector p by the formula

$$p \to P - C_n(\eta\cdot P, \varepsilon\cdot P, \varepsilon^*\cdot P)\eta \pm \frac{\Omega}{2\eta\cdot P}\eta, \tag{68}$$

where $\pm$ correspond respectively to the terms $d_{p,n}^-$ and $d_{p,n}^+$ with

$$C_n(\eta\cdot P, \varepsilon\cdot P, \varepsilon^*\cdot P) = \frac{1}{\eta\cdot P}\left(n\eta\cdot P - (n+1)\Omega + \frac{\Omega}{\Omega - \eta\cdot P}(\varepsilon^*\cdot P)(\varepsilon\cdot P)\right). \tag{69}$$

Expression (65) of the Green function is then rewritten

$$\begin{aligned}\widetilde{\mathbb{S}}^c(b,a) = (\mathrm{i})(\not P + m) \sum_{n=0}^{\infty} \int \frac{\mathrm{d}^4 P}{(2\pi)^4} \\ \times \exp\Big\{ \mathrm{i}(P - C_n\eta + n\eta)\cdot(x_b - x_a) \\ + \frac{\Omega(\varepsilon^*\cdot P)(\varepsilon\cdot P)}{(\Omega - \eta\cdot P)^2} \\ + \left(\frac{\sqrt{\Omega}\varepsilon\cdot P}{\omega}\mathrm{e}^{-\mathrm{i}\eta\cdot x_a} z_a - \frac{\sqrt{\Omega}\varepsilon^*\cdot P}{\omega}\mathrm{e}^{\mathrm{i}\eta\cdot x_a} z_a^*\right) \\ + \left(\frac{\sqrt{\Omega}\varepsilon^*\cdot P}{\Omega - \eta\cdot P}\mathrm{e}^{\mathrm{i}\eta\cdot x_b} z_b^* - \frac{\sqrt{\Omega}\varepsilon\cdot P}{\Omega - \eta\cdot P}\mathrm{e}^{-\mathrm{i}\eta\cdot x_b} z_b\right)\end{aligned}$$

$$- \mid z_b \mid^2 - \mid z_a \mid^2 \Big\} \frac{(z_b^*)^n (z_a)^n}{n!}$$

$$\times \Bigg\{ \frac{\exp\left(\frac{\mathrm{i}\Omega}{2\eta\cdot P}\eta\cdot(x_b - x_a)\right)}{P^2 - m^2 + \mathrm{i}\epsilon} \Bigg[\frac{1}{2}(1+S)$$

$$+ \left(\frac{\sqrt{\Omega}}{2\eta\cdot P}(\mathrm{e}^{-\mathrm{i}\eta\cdot x_b} - \mathrm{e}^{-\mathrm{i}\eta\cdot xa}) z_a - \frac{\varepsilon^*\cdot P}{2(\Omega - \eta\cdot P)} \right)$$

$$\times \not\varepsilon \not\eta$$

$$+ \left(\frac{\sqrt{\Omega}}{2\eta\cdot P}(\mathrm{e}^{\mathrm{i}\eta\cdot x_b} - \mathrm{e}^{\mathrm{i}\eta\cdot xa}) z_b^* + \frac{\varepsilon\cdot P}{2(\Omega - \eta\cdot P)} \right)$$

$$\times \not\varepsilon^* \not\eta \Bigg] + \frac{\exp\left(-\frac{\mathrm{i}\Omega}{2\eta\cdot P}\eta\cdot(x_b - x_a)\right)}{P^2 - m^2 + \mathrm{i}\epsilon}$$

$$\times \Bigg[\frac{1}{2}(1-S) + \frac{\varepsilon^*\cdot P}{2(\Omega - \eta\cdot P)} \not\varepsilon \not\eta$$

$$- \frac{\varepsilon\cdot P}{2(\Omega - \eta\cdot P)} \not\varepsilon^* \not\eta \Bigg] \Bigg\}. \tag{70}$$

In order to determine the wave functions, let us integrate over the energy P_0 by making use of the residue theorem where the contour of integration is closed into the upper half-plane. Next, we evaluate the integrations over the initial and final complex variable z via the identity

$$\int \frac{\mathrm{d}z^*\mathrm{d}z}{2\mathrm{i}\pi} \mathrm{e}^{-|z|^2} (z^*)^n (z)^m = \sqrt{n!}\sqrt{m!}\delta n, m. \tag{71}$$

Finally, let us proceed to the application of the operator projection ($\not\mathcal{P}+m$). As a consequence, the Green function related to the Dirac particle interacting with a quantized plane wave takes the symmetric form

$$\widetilde{\mathbb{S}}^{\mathrm{c}}(b,a) = -\frac{1}{2P_0} \sum_{s=\pm 1} \sum_{n=0}^{+\infty} \int \frac{\mathrm{d}^3 P}{(2\pi)^3}$$

$$\times \Bigg[- \not P - g \not A_b + m - \frac{g}{g^2h^2 - \eta\cdot P} P\cdot A_b \not\eta$$

$$+ \frac{(\varepsilon\cdot P)(\varepsilon^*\cdot P) g^2 h^2}{(g^2h^2 - \eta\cdot P)^2} \not\eta \Bigg]$$

$$\times \Bigg\{ (\cos\beta + \mathrm{i}S\sin\beta) D_P \mid n\rangle\langle n \mid D_P^\dagger$$

$$+ \frac{\mathrm{i}(\varepsilon^*\cdot P)}{g^2h^2 - \eta\cdot P} \not\varepsilon \not\eta D_P \mid n\rangle\langle n \mid D_P^\dagger \sin\beta$$

$$- \frac{\mathrm{i}(\varepsilon\cdot P)}{g^2h^2 - \eta\cdot P} \not\varepsilon^* \not\eta D_P \mid n\rangle\langle n \mid D_P^\dagger \sin\beta$$

$$- \frac{gh}{2\eta\cdot P} \left(\mathrm{e}^{-\mathrm{i}\eta\cdot x_b} - \mathrm{e}^{-\mathrm{i}\eta\cdot x_a}\right) \not\varepsilon \not\eta D_P \mid n\rangle\langle n \mid D_P^\dagger a \mathrm{e}^{\mathrm{i}\beta}$$

$$- \frac{gh}{2\eta\cdot P} \left(\mathrm{e}^{\mathrm{i}\eta\cdot x_b} - \mathrm{e}^{\mathrm{i}\eta\cdot x_a}\right) \not\varepsilon^* \not\eta a^+ D_P \mid n\rangle\langle n \mid D_P^\dagger \mathrm{e}^{\mathrm{i}\beta}$$

$$+ \frac{g^2h^2(\varepsilon^*\cdot P)}{(g^2h^2 - \eta\cdot P)\eta\cdot P} \left(\mathrm{e}^{-\mathrm{i}\eta\cdot(x_b - x_a)} - 1\right)$$

$$\times \not\varepsilon \not\eta D_P \mid n\rangle\langle n \mid D_P^\dagger \mathrm{e}^{\mathrm{i}\beta}$$

$$+ \frac{g^2h^2(\varepsilon\cdot P)}{(g^2h^2 - \eta\cdot P)\eta\cdot P} \left(\mathrm{e}^{-\mathrm{i}\eta\cdot(x_b - x_a)} - 1\right) \not\varepsilon^* \not\eta$$

$$\times D_P \mid n\rangle\langle n \mid D_P^\dagger \mathrm{e}^{\mathrm{i}\beta} \Bigg\}, \tag{72}$$

where $P_0 = (\mathbf{P}^2 + m^2)^{1/2}$,

$$D_P = \exp\left[-\frac{gh(\varepsilon^*\cdot P)}{g^2h^2 - \eta\cdot P} a^+ + \frac{gh(\varepsilon\cdot P)}{g^2h^2 - \eta\cdot P} a \right]$$

and

$$\beta = \frac{g^2h^2}{2\eta\cdot P}\eta\cdot(x_b - x_a).$$

As a last step, according to the procedure of [15] we get for the bound state contribution of the matrix Green function

$$\widetilde{\mathbb{S}}^{\mathrm{c}}(b,a) = -\frac{1}{2P_0} \sum_{s=\pm 1} \sum_{n=0}^{+\infty} \int \frac{\mathrm{d}^3 P}{(2\pi)^3}$$

$$\times \exp\left(\mathrm{i} \left(P - \frac{g^2h^2(\varepsilon^*\cdot P)(\varepsilon\cdot P)}{(g^2h^2 - \eta\cdot P)\eta\cdot P}\eta \right) \cdot (x_b - x_a) \right)$$

$$\times [- \not P - g \not A_b + m] \exp\Bigg(- \frac{gh}{g^2h^2 - \eta\cdot P}$$

$$\times \left((\varepsilon\cdot P) a \mathrm{e}^{-\mathrm{i}\eta\cdot x_b} + (\varepsilon^*\cdot P) a^+ \mathrm{e}^{\mathrm{i}\eta\cdot x_b} \right) \Bigg)$$

$$\times \exp\left(\mathrm{i}\eta\cdot(x_b - x_a) \left(n + 1 - \frac{1}{2}s \right) \frac{g^2h^2}{\eta\cdot P} \right)$$

$$\times \not\eta \chi_s \mid n\rangle\langle n \mid \chi_s^+ \gamma^0 \not\eta \exp\Bigg(\frac{gh}{g^2h^2 - \eta\cdot P}$$

$$\times \left((\varepsilon\cdot P) a \mathrm{e}^{-\mathrm{i}\eta\cdot x_a} + (\varepsilon^*\cdot P) a^+ \mathrm{e}^{\mathrm{i}\eta\cdot x_a} \right) \Bigg)$$

$$\times [- \not P - g \not A_a + m]$$

$$= \sum_{s=\pm 1} \sum_{n=0}^{+\infty} \int \mathrm{d}^3 P \Psi_{P,n,s}(x_b) \Psi_{P,n,s}^\dagger(x_a) \gamma^0, \tag{73}$$

where χ_s is an eigenfunction of the spin operator S : $S\chi_s = s\chi_s$, $s = \pm 1$.

Thus the suitably normalized wave functions describing the motion of the Dirac particle are

$$\Psi_{P,n,s}(x) = \frac{1}{2} \frac{1}{(2\pi)^{3/2}} \left(\frac{1}{(\eta\cdot P) P_0} \right)^{1/2}$$

$$\times \exp\left(\mathrm{i} \left(P - \frac{g^2h^2(\varepsilon^*\cdot P)(\varepsilon\cdot P)}{(g^2h^2 - \eta\cdot P)\eta\cdot P}\eta \right) \cdot x \right)$$

$$\times [- \not P - g \not A(\eta\cdot x) + m]$$

$$\times \exp\Bigg(- \frac{gh}{g^2h^2 - \eta\cdot P}$$

$$\times \left((\varepsilon \cdot P)a\mathrm{e}^{-\mathrm{i}\eta\cdot x} + (\varepsilon^* \cdot P)a^+\mathrm{e}^{\mathrm{i}\eta\cdot x}\right)\Bigg) \tag{74}$$

$$\times \exp\left(\mathrm{i}\eta \cdot x \left(n + 1 - \frac{1}{2}s\right)\frac{g^2h^2}{\eta \cdot P}\right) \not{\eta}\chi_s \mid n\rangle.$$

The results (67) and (74) are equivalent with those of [14].

3 Conclusion

In this paper, we have calculated, within the framework of the global path integral representation [13], the Green function of the relativistic electron interacting with the one-mode circularly polarized electromagnetic field of a quantized plane wave. We have shown that this result has been obtained in a natural way by elementary techniques without the use of the usual five-dimensional extension and multiplication with the matrix γ^5 [5]. The essential steps in the calculation are the choice of a convenient gauge (Lorentz gauge) and the introduction of two fermionic identities. The generated auxiliary fermionic variables decouple the free evolution of the spin from the spin–photon field coupling. Owing to this dimensional extension the classical equations of motion for the spin variables have appeared and the problem has thus become solvable. The energy spectrum and the correctly normalized wave functions of the bound states have been deduced. Finally, it is important to note that for the case of the problem of the relativistic spinning particle in the quantized plane wave field, the global projection method is preferable. The spin–photon field coupling terms in the action, which appear in the local projection method and complicate a lot the calculations, will consequently be reduced enormously.

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